On a special structured matrix problem

Cong Sun

Beijing University of Posts and Telecommunications
Technische Universität Dresden
Email: suncong86@bupt.edu.cn

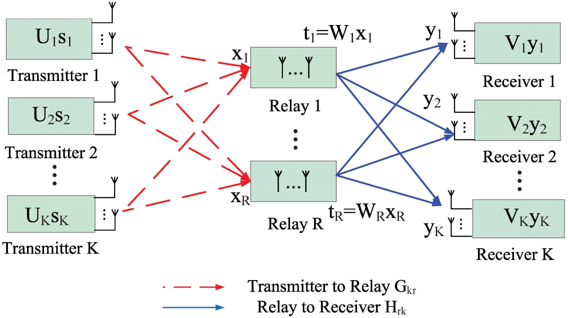
Joint work with Eduard Jorswieck and Yaxiang Yuan

Toulouse, Jan. 13th

Outline

1. Introduction
2. TSTINR approximation model
3. Multiple stream TSTINR model
4. Simulation results
5. Conclusion

Applications



- multiple user pairs, multiple relays, multiple antennas

Transmission process

Time slot 1: transmitters → relays

Time slot 2: relays → receivers

$$\tilde{\mathbf{y}}_k = \underbrace{\mathbf{V}_k^H \mathbf{T}_{kk}\mathbf{s}_k}_{\text{desired signal}} + \underbrace{\sum_{q\in\mathcal{K},q\neq k} \mathbf{V}_k^H \mathbf{T}_{kq}\mathbf{s}_q}_{\text{interference}} + \underbrace{\sum_{r\in\mathcal{R}} \mathbf{V}_k^H \mathbf{H}_{kr}\mathbf{W}_r\mathbf{n}_r + \mathbf{V}_k^H \mathbf{z}_k}_{\text{noise}} \qquad (1.1)$$

where $\mathbf{T}_{kq} = \sum_{r\in\mathcal{R}} \mathbf{H}_{kr}\mathbf{W}_r\mathbf{G}_{rq}\mathbf{U}_q$.

- Optimize:

−− **precoding matrices** $\{\mathbf{U}\}$

−− **decoding matrices** $\{\mathbf{V}\}$

−− **relay AF matrix** $\{\mathbf{W}\}$

Sum rate maximization

- Aim: maximize system sum rate under certain power constraints

$$R_{\text{sum}} = \frac{1}{2} \sum_{k \in \mathcal{K}} \log_2 \det(\mathbf{I}_{N_k} + \mathbf{F}_k^{-1} \mathbf{T}_{kk} \mathbf{T}_{kk}^H), \tag{1.2}$$

where $\mathbf{F}_k = \sum_{q \neq k, q \in \mathcal{K}} \mathbf{T}_{kq} \mathbf{T}_{kq}^H + \sum_{r \in \mathcal{R}} \sigma_r^2 \mathbf{H}_{kr} \mathbf{W}_r (\mathbf{H}_{kr} \mathbf{W}_r)^H + \mu_k^2 \mathbf{I}_{N_k}$.

☹ Difficult to optimize directly

☺ Approximation model to approximate sum rate maximization

TSTINR approach

- Total Signal to Total Interference plus Noise Ratio (TSTINR) definition:

$$\text{TSTINR} = \frac{P^S}{P^I + P^N} = \frac{\sum_{k\in\mathcal{K}} P_k^S}{\sum_{k\in\mathcal{K}}(P_k^I + P_k^N)}.$$

Theorem 2.1

For any precoder $\mathbf{U}_k, k \in \mathcal{K}$, *relay beamforming matrix* $\mathbf{W}_r, r \in \mathcal{R}$ *and any decoder* $\mathbf{V}_k$ *satisfying* $\mathbf{V}_k^H\mathbf{V}_k = \mathbf{I}_{d_k}, k \in \mathcal{K}$, *we have*

$$log_2[1 + TSTINR(\{\mathbf{U}\}, \{\mathbf{V}\}, \{\mathbf{W}\})] \leq R_{sum}(\{\mathbf{U}\}, \{\mathbf{W}\}).$$

TSTINR model

- With individual user and relay transmit power constraints:

$$\begin{aligned} \max_{\substack{\{\mathbf{U}\},\{\mathbf{V}\},\\ \{\mathbf{W}\}}} \quad & \text{TSTINR} = \frac{\sum_{k\in\mathcal{K}} P_k^S}{\sum_{k\in\mathcal{K}}(P_k^I + P_k^N)} \\ \text{s.t.} \quad & \mathbf{V}_k^H \mathbf{V}_k = \mathbf{I}_{d_k}, \|\mathbf{U}_k\|_F^2 \le p_k^U, k \in \mathcal{K}, \\ & \sum_{k\in\mathcal{K}} \|\mathbf{W}_r \mathbf{G}_{rk} \mathbf{U}_k\|_F^2 + \sigma_r^2 \|\mathbf{W}_r\|_F^2 \le p_r^R, r \in \mathcal{R}. \end{aligned} \tag{2.1}$$

- Problem reformulation:

$$\begin{aligned} \min_{\substack{\{\mathbf{U}\},\{\mathbf{V}\},\\ \{\mathbf{W}\}}} \quad & f(\{\mathbf{U}\},\{\mathbf{V}\},\{\mathbf{W}\}; C) = C(P^I + P^N) - P^S \\ \text{s.t.} \quad & \ldots \end{aligned} \tag{2.2}$$

Problem reformulation

- Update strategy of the parameter C:

$$C = \frac{P^S(\{\mathbf{U}\}, \{\mathbf{V}\}, \{\mathbf{W}\})}{P^I(\{\mathbf{U}\}, \{\mathbf{V}\}, \{\mathbf{W}\}) + P^N(\{\mathbf{U}\}, \{\mathbf{V}\}, \{\mathbf{W}\})}. \tag{2.3}$$

Theorem 2.2

If the objective function of (2.2) has sufficient reduction in each iteration and C is updated as (2.3), then the objective function of (2.1), TSTINR, is monotonically increasing. Any KKT point of (2.2) is also a KKT point of (2.1).

Alternating direction algorithm

- Decoding subproblem:

$$\begin{aligned} \min_{\mathbf{X}\in\mathbb{C}^{N_k\times d_k}} \quad & \mathrm{tr}(\mathbf{X}^H\mathbf{A}_0\mathbf{X}) \\ \text{s.t.} \quad & \mathbf{X}^H\mathbf{X} = \mathbf{I}_{d_k} \end{aligned} \tag{2.4}$$

  – Eigenvalue problem

- Equivalent subproblem for relay AF matrix $\mathbf{W}_r$:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{C}^{L_r^2\times 1}} \quad & \bar{f}(\mathbf{x}) = \mathbf{x}^H\mathbf{B}_1\mathbf{x} + \mathbf{b}^H\mathbf{x} + \mathbf{x}^H\mathbf{b} \\ \text{s.t.} \quad & \mathbf{x}^H\mathbf{B}_2\mathbf{x} \leq \eta_1, \end{aligned} \tag{2.5}$$

  where $\mathbf{B}_2 \succ 0$.

  – Trust region subproblem

Alternating iteration algorithm

- Equivalent subproblem for precoder $\mathbf{U}_k$:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^{n\times 1}} \quad & f(\mathbf{x}) = \mathbf{x}^T\mathbf{Q}_0\mathbf{x} + 2\mathbf{g}^T\mathbf{x} \\ \text{s.t.} \quad & C_i(\mathbf{x}) = \mathbf{x}^T\mathbf{Q}_i\mathbf{x} - 1 \leq 0, i = 1,\dots,m, \end{aligned} \tag{2.6}$$

  - $\mathbf{Q}_0$ indefinite; $\mathbf{Q}_i \succeq 0, i = 1,\dots,m$
  - Nonconvex Quadratic Constrained Quadratic Programming (QCQP)

Idea of feasible shrinkage method

- Initial point:

$$\min_{\mathbf{x}\in\mathbb{R}^{n\times 1}} \quad f(\mathbf{x}) = \mathbf{x}^T\mathbf{Q}_0\mathbf{x} + 2\mathbf{g}^T\mathbf{x}$$
$$\text{s.t.} \quad \mathbf{x}^T\sum_{i=1}^{m}\mathbf{Q}_i\mathbf{x} \le 1 \tag{2.7}$$

 – The optimal solution of (2.7) is a feasible solution of (2.6).

 For any $\mathbf{x}$ as the feasible solution of (2.7),

$$\mathbf{x}^T\mathbf{Q}_i\mathbf{x} \le \mathbf{x}^T\sum_{i=1}^{m}\mathbf{Q}_i\mathbf{x} \le 1, i = 1, \ldots, m.$$

Idea of feasible shrinkage method

- Subproblem in each iteration (with the iterative point $\mathbf{x}_0$):

$$\begin{aligned} \min_{\mathbf{d}\in\mathbb{C}^{n\times 1}} \quad & \mathbf{d}^H\mathbf{Q}_0\mathbf{d} + \bar{\mathbf{g}}^H\mathbf{d} + \mathbf{d}^H\bar{\mathbf{g}} \\ \text{s.t.} \quad & \mathbf{d}^H\bar{\mathbf{Q}}\mathbf{d} \leq 1, \end{aligned} \tag{2.8}$$

  where $\bar{\mathbf{g}} = \mathbf{g} + \mathbf{Q}_0\mathbf{x}_0$, $\bar{\mathbf{Q}} = \sum_{i=1}^m \rho_i(\mathbf{Q}_i + \rho_i\mathbf{Q}_i\mathbf{x}_0\mathbf{x}_0^H\mathbf{Q}_i)$ and

$$\rho_i = \frac{2}{1 - \mathbf{x}_0^H\mathbf{Q}_i\mathbf{x}_0}, i = 1, \ldots, m.$$

- Subproblem idea: weighted quadratic approximation

Feasible shrinkage method

Lemma 2.1

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ *be a positive semi-definite symmetric matrix and* $\mathbf{y} \in \mathbb{R}^{n\times 1}$ *is a vector, such that* $\mathbf{y}^T\mathbf{A}\mathbf{y} < 1$, *then*

$$\mathcal{B} = \{\mathbf{x} | (\mathbf{x}-\mathbf{y})^T(\mathbf{A}+\rho\mathbf{A}\mathbf{y}\mathbf{y}^T\mathbf{A})(\mathbf{x}-\mathbf{y}) \leq \frac{1}{\rho}\}$$

is the subset of $\mathcal{A} = \{\mathbf{x}|\mathbf{x}^T\mathbf{A}\mathbf{x} \leq 1\}$, *where* $\rho = \frac{2}{1-\mathbf{y}^T\mathbf{A}\mathbf{y}}$.

Theorem 2.3

If $\mathbf{x}_0$ *is an interior feasible point of (2.6) and* $\mathbf{d}_0$ *is the optimal solution of (2.8), then* $\mathbf{x}_0 + \mathbf{d}_0$ *is feasible for problem (2.6).*

Algorithm 1 (Feasible shrinkage algorithm)

Step 1 *Initialization: given stopping parameters* ϵ_0 *and* ϵ_1*; solve (2.7) and set its optimal solution as* $\mathbf{x}_0$*.*

Step 2 *With the feasible iterative point* $\mathbf{x}_0$*, solve subproblem (2.8).*

Step 3 *Update the iterative point with* $\mathbf{d}_0$*, the solution of (2.8), as* $\mathbf{x}_0 := \mathbf{x}_0 + \mathbf{d}_0$*.*

Step 4 *If* $||\mathbf{d}_0||_2 < \epsilon_1$*, or there exists* i *such that* $1 - \mathbf{x}_0^T \mathbf{Q}_i \mathbf{x}_0 < \epsilon_0$*, stop and output* $\mathbf{x}_0$*. Otherwise go back to Step 2.*

- feasible shrinkage method + nonconvex SQP
- interior point $\longrightarrow$ boundary point
- a good feasible solution $\longrightarrow$ KKT point

Algorithm 2 (Algorithm for TSTINR model)

Step 1 *Set initial value of* $\mathbf{U}_k, k \in \mathcal{K}$ *and* $\mathbf{W}_r, r \in \mathcal{R}$. $C = 1$.

Step 2 *Update decoder* $\mathbf{V}_k$ *by solving (2.4),* $k \in \mathcal{K}$.

Step 3 *Update relay beamforming matrix* $\mathbf{W}_r$ *by solving (2.5),* $r \in \mathcal{R}$.

Step 4 *Update precoder* $\mathbf{U}_k$ *by solving (2.6),* $k \in \mathcal{K}$.

Step 5 *Update* C *as* $C := \frac{P^S}{P^I + P^N}$. *Go back to Step 2. Iterate until convergence.*

- ☺ The objective function TSTINR converges
- ☹ No guarantee to converge to a KKT point

Analysis of precoding subproblem

- Precoding subproblem:

$$\begin{aligned} \min_{\mathbf{X}\in\mathbb{C}^{M_k\times d_k}} \quad & \mathrm{tr}(\mathbf{X}^H\mathbf{D}_0\mathbf{X}) \\ \text{s.t.} \quad & \mathrm{tr}(\mathbf{X}^H\mathbf{D}_r\mathbf{X}) \le \eta_r,\ r\in\mathcal{R}. \end{aligned} \tag{3.1}$$

Here $\mathbf{D}_r \geq 0, \eta_r \geq 0, r \in \mathcal{R}$

Theorem 3.1

The precoding subproblem (3.1) always has a rank one optimal solution.

- Observation: rank one precoding matrices
  linearly dependent columns

Multiple stream model

$$\max_{\{\mathbf{U}\},\{\mathbf{V}\},\{\mathbf{W}\}} \quad \text{TSTINR} = \frac{P^S}{P^I + P^N} = \frac{\sum_{k\in\mathcal{K}} P_k^S}{\sum_{k\in\mathcal{K}}(P_k^I + P_k^N)} \tag{3.2}$$

$$\text{s.t.} \quad \mathbf{U}_k^H\mathbf{U}_k = \frac{p_k^U}{d_k}\mathbf{I}_{d_k}, \leftarrow \text{Different from (2.1)}$$

$$\mathbf{V}_k^H\mathbf{V}_k = \mathbf{I}_{d_k}, k \in \mathcal{K},$$

$$\sum_{k\in\mathcal{K}} \|\mathbf{W}_r\mathbf{G}_{rk}\mathbf{U}_k\|_F^2 + \sigma_r^2\|\mathbf{W}_r\|_F^2 \le p_r^R, r \in \mathcal{R}$$

- Precoding subproblem:

$$\min_{\mathbf{X}\in\mathbb{C}^{M_k\times d_k}} \quad \text{tr}(\mathbf{X}^H\mathbf{D}_0\mathbf{X}) \tag{3.3a}$$

$$\text{s.t.} \quad \mathbf{X}^H\mathbf{X} = \rho\mathbf{I}_{d_k}, \tag{3.3b}$$

$$\text{tr}(\mathbf{X}^H\mathbf{D}_r\mathbf{X}) \le \eta_r, r \in \mathcal{R} \tag{3.3c}$$

– $\mathbf{D}_0$ indefinite; $\mathbf{D}_r \succeq 0$, $r \in \mathcal{R}$

– Nonconvex QCQP with orthogonality constraint

Dual problem

- Dual Problem of (3.3):

$$\begin{aligned}\min_{\theta_r, r\in\mathcal{R}} \quad & h(\theta_1, \theta_2, \ldots, \theta_R) \\ & = \sum_{r\in\mathcal{R}} \theta_r \eta_r - \min_{\mathbf{X}^H\mathbf{X}=\rho\mathbf{I}_{d_k}} \mathrm{tr}[\mathbf{X}^H(\mathbf{D}_0 + \sum_{r\in\mathcal{R}} \theta_r \mathbf{D}_r)\mathbf{X}] \\ \text{s.t.} \quad & \theta_r \geq 0, r \in \mathcal{R}. \end{aligned} \tag{3.4}$$

Lagrangian multiplier: $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_R)^T$

– Suppose $\rho = 1$

✓ Projected subgradient method

Projected subgradient method

- Subgradient of $h(\boldsymbol{\theta})$: $\mathbf{y} = (y_1, y_2, \ldots, y_R)^T$,

$$y_r = \frac{\partial h}{\partial \theta_r} = \eta_r - \mathrm{tr}(\mathbf{X}^H \mathbf{D}_r \mathbf{X})$$

  – $\mathbf{X}$: eigenvectors of the smallest d_k eigenvalue of $\mathbf{D}_0 + \sum_{r \in \mathcal{R}} \theta_r \mathbf{D}_r$

Ref Rudisill, "Derivatives of Eigenvalues and Eigenvectors for a General Matrix," AIAA J., 1974

Algorithm 3 (Algorithm for subproblem (3.3))

Step 1 *Randomly set initial point* $\boldsymbol{\theta} \geq \mathbf{0}$. *Set the stopping criterion* $\epsilon \geq 0$. $j = 1$.

Step 2 *Update the iterative point:* $\boldsymbol{\theta}^{j+1} = (\boldsymbol{\theta}^j - \frac{1}{j}\mathbf{y}^j)_+$. $j := j + 1$.

Step 3 *If* $\|\boldsymbol{\theta}^{j+1} - (\boldsymbol{\theta}^j\|_2 < \epsilon$, *stop and output* $\boldsymbol{\theta}^* = \boldsymbol{\theta}^j$ *as the solution of (3.4) and* $\mathbf{X}^* = \nu_{\min}^{d_k}(\mathbf{D}_0 + \sum_{r \in \mathcal{R}} \theta_r^* \mathbf{D}_r)$ *as the solution of (3.3); otherwise, go back to Step 2.*

Optimality

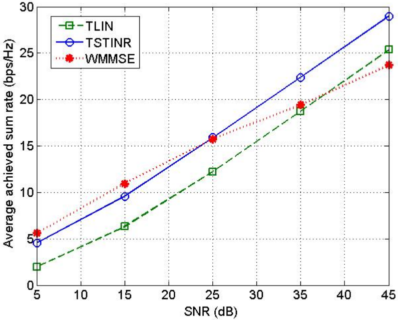
Theorem 3.2

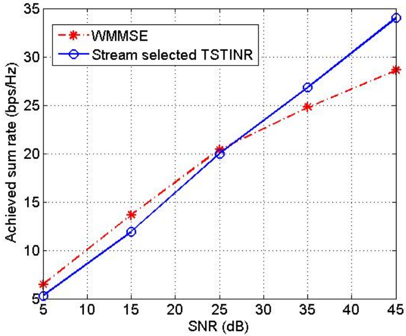
If $\boldsymbol{\theta}^*$ is the optimal solution of the dual problem (3.4), then $\mathbf{X}^*$ is a feasible point of (3.3). Furthermore, if $h(\boldsymbol{\theta})$ is smooth at $\boldsymbol{\theta} = \boldsymbol{\theta}^*$, then $\mathbf{X}^*$ is the optimal solution of (3.3).

Simulation settings

- Each element of $\mathbf{G}_{rk}$ and $\mathbf{H}_{kr}, k \in \mathcal{K}, r \in \mathcal{R}$: $\mathcal{CN}(0, 1)$
- $\sigma_1^2 = \sigma_2^2 = \sigma^2 = 1$, SNR= $\frac{p_0^U}{\sigma^2} = \frac{p_0^R}{\sigma^2}$.
- Initialization: randomly generated initial points;
  penalty parameter $C = 1$.
- FS: $\epsilon_0 = 10^{-4}, \epsilon_0 = 10^{-8}$; SQP: $\epsilon = 10^{-8}$, $u = 1$, $\eta = 0.1$.
- 100 random realization of different channel coefficients for each plotted point

Comparison of different approaches





- Left: single stream; Right: multiple stream

√ TSTINR achieves highest sum rate under medium to high SNR

Ref Truong, Sartori and Heath, "Cooperative Algorithms for MIMO Amplify-and-Forward Relay Networks," *IEEE Trans. Signal Process.*, 2013.

Comparison of different algorithms

SNR value	5dB	15dB	25dB	35dB	45dB
H-Sum rate (bps/Hz)	8.4579	14.1468	18.8045	23.4674	28.1119
C-Sum rate (bps/Hz)	8.4580	14.1468	18.8201	23.5132	28.1319
H-Time (s)	5.4138	5.5940	6.0208	5.9795	6.6703
C-Time (s)	106.3531	105.7086	106.3560	106.2938	107.1608

- convex subproblem: FS+ nonconvex SQP (H) **vs** CVX (C)

$\surd$ Significantly less computing time with almost the same result

Conclusion

For the sum rate maximization problem in wireless communications:

- Propose efficient algorithms for the TSTINR model
  (Algorithm for fractional optimization, special structured QCQP)

- Propose the multiple stream model and an efficient algorithm
  (QCQP with orthogonal constraint)

Thank you for listening!